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Symmetries in Quantum Logics

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Abstract

A symmetry in the quantum logic (L, M) is defined as a pair of bijections $\alpha : L \to L$ and $\nu : M \to M$ such that the probabilities are preserved. Some properties of the symmetries are investigated.

1. *Introduction*

A symmetry of a physical system is intuitively a transformation of the system, leaving all physically significant features invariant.

In the quantum logic approach, the quantum theory is introduced in terms of the set of propositions L of a physical system and the set of states M of that system. Each state of M defines a probability measure on L. We shall define a symmetry to be a pair of bijections $\alpha : L \to L$ and $\nu : M \to M$ such that $\nu(m)(\alpha(a)) = m(a)$ for each $m \in M$ and $a \in L$. Thus the probabilities are preserved by a symmetry. This definition is analogous to the definition of symmetrices in C^* algebras introduced by Roberts and Roepstorff (1969). We shall analyze the properties of a symmetry in a system (L, M).

2. *Definitions and Notation*

Let L be a partially ordered set with first and last elements 0, 1, respectively, which is closed under a complementation $a \mapsto a'$ satisfying

(i) $(a')' = a$
(ii) $a \leq b$ implies $b' \leq a'$

We denote the least upper bound and greatest lower bound of $a, b \in L$, if they exist, by $a \vee b$ and $a \wedge b$, respectively, and assume

(iii) $a \vee a' = 1$ for all $a \in L$

We say that $a, b \in L$ are disjoint and write $a \perp b$ if $a \leq b'$. We say that $a, b \in L$ are compatible and write $a \leftrightarrow b$ if there exist mutually disjoint elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. We call L a logic if it also satisfies

(iv) $\vee a_i \in L$ for any disjoint sequence $(a_i) \subset L$
(v) if $a, b, c \in L$ are mutually compatible, then $a \leftrightarrow b \vee c$

The set of propositions of a physical system is supposed to be a logic (Mackey, 1963; Varadarajan, 1962, 1968; Gudder, 1967).

A state is a non-negative function on L satisfying

(i) $m(1) = 1$
(ii) $m(\vee a_i) = \Sigma m(a_i)$ for any disjoint sequence $(a_i) \subset L$.

A set M of states is full if $m(a) \leq m(b)$ for all $m \in M$ imply $a \leq b, a, b \in L$. A logic with a full set of states has the orthomodularity property

$$a \leq b \text{ implies } b = a \vee (b \wedge a')$$

A state $m \in M$ is pure if it cannot be written in the form

$$m = cm_1 + (1 - c)m_2, \qquad \text{where } 0 < c < 1 \text{ and } m_1, m_2 \in M$$

Let $P \subset M$ be the set of all pure states. The set of states of a physical system is usually supposed to be closed under countable convex combinations, i.e., $m_i \in M, i = 1, 2, \ldots$ imply $\Sigma\, t_i m_i \in M$ for any sequence (t_i) of real numbers such that $0 \leq t_i \leq 1$ and $\Sigma t_i = 1$.

We shall call the couple (L, M), where L is a logic and M is a convex full set of states, the quantum logic.

Let L and L' be orthomodular logics. The map $T: L \to L'$ is a σ homomorphism if

(i) $T(0) = 0$
(ii) $T(a') = T(a)'$ for any $a \in L$
(iii) $T(\vee a_i) = \vee(T(a_i))$ for any disjoint sequence $(a_i) \subset L$

The one-to-one σ homomorphism of L onto L is an automorphism. Automorphisms in special types of logics were treated by Emch and Piron (1963), Dvurečenskij (1976), and Kruszyński (1976).

Let (L, M) be a quantum logic. An observable x is a σ homomorphism from the Borel sets $B(R)$ of the real line R into L. A collection of observables $\{x_\lambda : \lambda \in \Lambda\}$ is simultaneous if $x_\lambda(E) \leftrightarrow x_\mu(F)$ for all $E, F \in B(R)$ and $\lambda, \mu \in \Lambda$. If x is an observable and u a Borel function on R, we define the observable $u(x)$ by $u(x)(E) = x(u^{-1}(E))$ for all $E \in B(R)$. More generally, if ψ is an n-dimensional Borel function and $u_1, u_2, \ldots, u_n$ are Borel functions on R, we define the observable $\psi(u_1(x), \ldots, u_n(x))$ by

$$\psi(u_1(x), \ldots, u_n(x))(E) = x\{\omega : \psi(u_1(\omega), \ldots, u_n(\omega)) \in E\}$$

for all $E \in B(R)$.

The spectrum $\sigma(x)$ of an observable x is the smallest closed set E such that

$x(E) = 1$. An observable is bounded if $\sigma(x)$ is bounded. The expectation of an observable x in the state m is

$$m(x) = \int \lambda m[x(d\lambda)]$$

if the integral exists.

An observable x is a proposition observable if $\sigma(x) \subset \{0, 1\}$. The following statements are equivalent (Mackey, 1963):

(i) x is a proposition observable
(ii) x is an indicator function of an observable
(iii) $x^2 = x$

Let X be the set of all observables and X_L the set of all proposition observables on L. We can define a partial ordering on X_L by setting $x \leq y$ if $m(x) \leq m(y)$ for all $m \in M$ and an orthocomplementation by setting $x^\perp = f(x)$, where $f(t) = 1 - t, t \in R$. To each $a \in L$ there is an observable $x_a \in X_L$ such that $x_a(\{1\}) = a$. It can be easily seen that the map $a \mapsto x_a$ from L onto X_L is an isomorphism.

3. *Properties of a Symmetry*

Definition 1. Let (L, M) be a quantum logic. A pair of bijections $\alpha : L \to L$ and $\nu : M \to M$ will be called a symmetry if $\nu(m)(\alpha(a)) = m(a)$ for all $a \in L, m \in M$.

Proposition 1. If $\alpha : L \to L$ and $\nu : M \to M$ arise from a symmetry, then α is an automorphism of L and ν preserves countable convex combinations in M.

Proof. As $\alpha : L \to L$ and $\nu : M \to M$ are bijections, we have $\alpha[L] = \{\alpha(a) : a \in L\} = L$ and $\nu[M] = \{\nu(m) : m \in M\} = M$. The inverse maps α^{-1} and ν^{-1} exist and arise from a symmetry as well. From $m(\alpha(0)) = \nu^{-1}(m)(\alpha^{-1}\alpha(0)) = \nu^{-1}(m)(0) = 0$ for all $m \in M$ we get $\alpha(0) = 0$. From $1 = m(a \vee a') = m(a) + m(a')$ for all $m \in M$ we have $m(\alpha(a)') = 1 - m(\alpha(a)) = 1 - \nu^{-1}(m)(a) = \nu^{-1}(m)(a') = m(\alpha(a'))$, that is $\alpha(a)' = \alpha(a')$. Now $a \leq b, a, b \in L$ implies by the orthomodularity property $m(a) \leq m(b)$ for all $m \in M$. From this it follows $m(\alpha(a)) = \nu^{-1}(m)(a) \leq \nu^{-1}(m)(b) = m(\alpha(b))$ for all $m \in M$, that is $a \leq b$ implies $\alpha(a) \leq \alpha(b)$. Let $(a_i) \subset L$ be a sequence of mutually disjoint elements. Then $a_i \leq a_j'$ implies $\alpha(a_i) \leq \alpha(a_j') = \alpha(a_j)'$. Thus we get $m(\vee\alpha(a_i)) = \Sigma m(\alpha(a_i)) = \Sigma\nu^{-1}(m)(a_i) = \nu^{-1}(m)(\vee a_i)$ for all $m \in M$, i.e., $\vee\alpha(a_i) = \alpha(\vee a_i)$. We have shown that α is an automorphism of L. Now let $m \in M$, $m = \Sigma t_i m_i$, where $m_i \in M$ and $0 \leq t_i \leq 1$, $\Sigma t_i = 1$. Then for any $a \in L$ $\nu(m)(a) = m(\alpha^{-1}(a)) = \Sigma t_i m_i(\alpha^{-1}(a)) = \Sigma t_i \nu(m_i)(a)$, that is $\nu(m) = \Sigma t_i \nu(m_i)$.

Corollary 1. Let $\nu : M \to M$ arise from a symmetry. Then $m \in P$ implies $\nu(m) \in P$.

Corollary 2. Let $\alpha : L \to L$ arise from a symmetry. Then $a \leftrightarrow b$ implies $\alpha(a) \leftrightarrow \alpha(b)$.

Let $\alpha : L \to L$ be an automorphism. We shall define the map $\bar{\alpha} : X \to X$ by $\bar{\alpha}(x)(E) = \alpha(x(E))$ for all $E \in B(R)$. It can be easily seen that $\bar{\alpha}$ is a bijection.

Proposition 2. Let (α, ν) be a symmetry. Then $\nu(m)(\bar{\alpha}(x)) = m(x)$, in the sense that if either side of the equation exist, so does the other and the equality holds.

Proof.

$$m(x) = \int \lambda m(x(d\lambda)) = \int \lambda \nu(m)[\alpha(x(d\lambda))] = \int \lambda \nu(m)[\bar{\alpha}(x)(d\lambda)] = \nu(m)(\bar{\alpha}(x))$$

Proposition 3. Let $\alpha : L \to L$ be an automorphism. Then for each observable x and each Borel function u on R $\bar{\alpha}(u(x)) = u(\bar{\alpha}(x))$.

Proof.

$$\bar{\alpha}(u(x))(E) = \alpha[u(x)(E)] = \alpha[x(u^{-1}(E))] = \bar{\alpha}(x)[u^{-1}(E)] = u(\bar{\alpha}(x))(E)$$

Proposition 4. Let (L, M) be a quantum logic, X the set of all observables, and X_L the set of all proposition observables on L. Let $\tau : X \to X$ be a bijection such that $\tau(f(x)) = f(\tau(x))$ for all $x \in X$ and all Borel functions f on R. Then the map $\alpha : L \to L$ defined by $\alpha(a) = \tau(x_a)(\{1\})$ is an automorphism.

Proof. Let $f(t) = t^2, t \in R$, then $f(\tau(x)) = \tau(f(x))$ implies that $\tau(x^2) = [\tau(x)]^2$. For $x \in X_L$ we have $x^2 = x$, so that $\tau(x^2) = \tau(x) = [\tau(x)]^2$. On the other hand, if $x \neq x^2$, then $\tau(x) \neq \tau(x^2) = [\tau(x)]^2$, as τ is one-to-one. Thus we have shown that $\tau[X_L] = \tau^{-1}[X_L] = X_L$. Let us set $\alpha(a) = \tau(x_a)(\{1\})$; then α is a one-to-one map of L onto L. We have to show that (i) $\alpha(1) = 1$, (ii) $\alpha(a') = \alpha(a)'$ and (iii) $\alpha(\vee a_i) = \vee\alpha(a_i)$ for any disjoint sequence $(a_i) \subset L$. To show (i) let x_1 be such that $x_1(\{1\}) = 1$. If x is any observable, then $I_R(x)(\{1\}) = x[I_R^{-1}(\{1\})] = x(R) = 1$, where I_R is the indicator function of R. Thus $I_R(x) = x_1$ for any $x \in X$. $\tau(I_R(x)) = I_R(\tau(x))$ implies $\tau(x_1) = x_1$, that is $\alpha(1) = \tau(x_1)(\{1\}) = x_1(\{1\}) = 1$. To show (ii) let $f(t) = 1 - t, t \in R$. Then $f(x_a)(\{1\}) = x_a(f^{-1}\{1\}) = x_a(\{0\}) = x_{a'}(\{1\})$, i.e., $f(x_a) = x_{a'}$ and $\alpha(a') = \tau(x_{a'})(\{1\}) = \tau(f(x_a))(\{1\}) = f(\tau(x_a))(\{1\}) = \tau(x_a)(\{0\}) = [\tau(x_a)(\{1\})]' = \alpha(a)'$. To show (iii), let $(a_i) \subset L$ be such that $a_i \leq a_j'$ for all $i \neq j, i, j = 1, 2, \ldots$. Clearly, $x_{a_i} \leftrightarrow x_{a_j}$ for all $i, j = 1, 2, \ldots$. Then by (Gudder, 1967, Theorem 2.4), there is an observable x and Borel functions u_i such that $x_{a_i} = u_i(x)$. Let us observe that

$$m(x_{\vee_1^n a_i}) = m(x_{\vee_1^n a_i}(\{1\})) = m(\bigvee_1^n a_i) = \sum_1^n m(a_i) = \sum_1^n m(x_{a_i})$$

for all $m \in M$, so that the observable $\Sigma_1^n x_{a_i}$ exists and equals $x_{\vee_1^n a_i}$ (Gudder, 1966). Let us set $\Sigma_1^n x_{a_i} = \psi(x_{a_1}, \ldots, x_{a_n}) = \psi(u_1, \ldots, u_n) = \varphi(x)$. From $\tau(\varphi(x)) = \varphi(\tau(x))$ it follows that $\tau(\Sigma_1^n x_{a_i}) = \tau(\psi(u_1, \ldots, u_n)(x)) = \psi(u_1, \ldots, u_n)(\tau(x)) = \Sigma_1^n u_i(\tau(x)) = \Sigma_1^n \tau(u_i(x)) = \Sigma_1^n \tau(x_{a_i})$. Now we can show that α preserves the order. Let $a \leq b$, then $b = a \vee c$, where $a \leq c'$ by the orthomodularity property. Then $x_b = x_a + x_c$ implies $\tau(x_b) = \tau(x_a) + \tau(x_c)$, i.e., $\tau(x_a) \leq \tau(x_b)$, which implies $m[\tau(x_a)(\{1\})] \leq m[\tau(x_b)(\{1\})]$ for all $m \in M$, that is, $\alpha(a) \leqslant \alpha(b)$. From the existence of τ^{-1} such that $\tau^{-1}(f(x)) = f(\tau^{-1}(x))$ (indeed, $\tau[f(\tau^{-1}(x))] = f[\tau\tau^{-1}(x)]$ implies $f(\tau^{-1}(x)) = \tau^{-1}(f(x))$ there follows the existence of α^{-1} also preserving the order. Then $a_i \leq \bigvee_1^\infty a_i, i = 1, 2, \ldots$ imply $\alpha(a_i) \leq \alpha(\bigvee_1^\infty a_i), i = 1, 2, \ldots$. Let $b \in L$ be such that $\alpha(a_i) \leq b$ for $i = 1, 2, \ldots$. Then $a_i \leq \alpha^{-1}(b)$ $i = 1, 2, \ldots$ imply $\bigvee_1^\infty a_i \leq \alpha^{-1}(b)$, that is, $\alpha(\bigvee_1^\infty a_i) \leq b$. Thus we have shown that $\alpha(\bigvee_1^\infty a_i) = \bigvee_1^\infty \alpha(a_i)$.

Now we shall investigate the case in which one of the maps $\alpha : L \to L$ and $\nu : M \to M$ is sufficient to define a symmetry.

> *Proposition* 5. Let (L, S) be a quantum logic, where S is the set of all states on L. If $\alpha : L \to L$ is an automorphism, then there exists a bijection $\nu : S \to S$ such that (α, ν) is a symmetry.

Proof. Let us set $\nu(m)(a) = m(\alpha^{-1}(a))$ for all $m \in S, a \in L$. It is easy to check that $a \mapsto \nu(m)(a)$ is a probability measure on L, so that $\nu(m) \in S$ and $\nu : S \to S$ is a bijection. □

Let $\mathscr{P}$ be the set of all probability measures on R and let $\mathrm{Hom}(S, \mathscr{P})$ be the set of all convex homomorphisms on S into $\mathscr{P}$. Then a set of observables X on the logic L is said to be total if to each element of $\mathrm{Hom}(S, \mathscr{P})$, there corresponds a unique observable in X (Kronfli, 1970). That is, if $\beta \in \mathrm{Hom}(S, \mathscr{P})$, then there is an $x \in X$ such that $\beta(p)(E) = p(x(E))$ for all $p \in S, E \in B(R)$.

> *Proposition* 5. Let (L, S) be a quantum logic such that the set of all observables X is total. Then to each convex isomorphism $\nu : S \to S$ there is an automorphism $\alpha : L \to L$ such that (α, ν) is a symmetry.

Proof. Let $\nu : S \to S$ be a convex isomorphism. Then for $x \in X$, the map $\beta : m \mapsto \nu(m)\ (x(\cdot))$ is a convex homomorphism of S into $\mathscr{P}$. From the totality of X it follows that there is a $y \in X$ such that $\nu(m)(x(E)) = m(y(E))$ for all $m \in S$ and $E \in B(R)$. Let us set $y = \tau^{-1}(x)$. Then $x \mapsto \tau^{-1}(x)$ maps X onto X and is one-to-one since ν is an isomorphism. Let f be any Borel function on R. Then $m[\tau^{-1}(f(x))(E)] = \nu(m)\ [f(x)(E)] = \nu(m)\ [x(f^{-1}(E))] = m[\tau^{-1}(x)(f^{-1}(E))] = m[f(\tau^{-1}(x))(E)]$ for all $m \in M, E \in B(R)$, so that $\tau^{-1}(f(x)) = f(\tau^{-1}(x))$. By Proposition 4, there is an automorphism α^{-1} of L such that $\tau^{-1}(x_a)(\{1\}) = \alpha^{-1}(a)$ for all $a \in L$. Then $\nu(m)(\alpha(a)) = \nu(m)[(\tau(x_a))(\{1\})] = m[\tau^{-1}(\tau(x_a))(\{1\})] = m(x_a(\{1\})) = m(a)$.

4. *Symmetries and the Superposition Principle*

In this sequel, we shall use the stronger form of quantum logics which was considered by Pulmannová (1976).

Let L be a logic, M a set of states on L, and P the set of all pure states in M. If $a \in L, m \in P$, we define $P_a = \{m \in P : m(a) = 1\}$, $L_m = \{a \in L : m(a) = 1\}$. We shall suppose that the system (L, M) satisfies the following:

(i) $P_a \subset P_b$ implies $a \leq b$
(ii) $L_{m_1} \subset L_{m_2}$ implies $m_1 = m_2$

It is easy to check that M is a full set of states on L. Indeed, let $m(a) \leq m(b)$ for all $m \in M$, then $m(a) = 1$ implies $m(b) = 1$ for all $m \in P$, so that $P_a \subset P_b$, which implies $a \leq b$.

We recall that $m_0 \in M$ is a superposition of the states $p, q \in M$ if $p(a) = 0$, $q(a) = 0$ imply $m_0(a) = 0$.

A set $S \subset P$ is said to be closed under superpositions if it contains every pure superposition of any pair of its elements. If S is not closed under superpositions, we denote $\Lambda(S)$ the smallest subset of P closed under superpositions and containing S.

We say that the superposition principle holds in (L, M) if there is an $r \in \Lambda(\{p, q\}), r \neq p, r \neq q$ for any pair p, q in $P, p \neq q$.

The set $S \subset P$ is a sector if (i) $S = \Lambda(S)$, (ii) if $p, q \in S$, then there is an $r \in \Lambda(\{p, q\}), r \neq p, r \neq q$, (iii) if $q \in P, q \notin S$, then $\Lambda(\{p, q\}) = \{p, q\}$ for any $p \in S$.

Proposition 7. If (α, ν) arise from a symmetry, then $\nu[\Lambda(S)] = \Lambda(\nu[S])$ for any $S \subset P$.

Proof. From Corollary 1 it follows that $\nu[\Lambda(S)] = \{\nu(p) : p \in \Lambda(S)\} \subset P$. We shall first show that $\nu[\Lambda(S)]$ is closed under superpositions. Let p, $q \in \nu[\Lambda(S)]$ and let r be a superposition of p, q, that is, let $p(a) = 0, q(a) = 0$ imply $r(a) = 0$. Let $\nu^{-1}(p)(b) = 0$ and $\nu^{-1}(q)(b) = 0$, then $p(\alpha(b)) = 0$ and $q(\alpha(b)) = 0$, which imply $r(\alpha(b)) = 0$, i.e., $\nu^{-1}(r)(b) = 0$. That is, $\nu^{-1}(r)$ is a superposition of $\nu^{-1}(p)$ and $\nu^{-1}(q)$. But $\nu^{-1}(p), \nu^{-1}(q) \in \Lambda(S)$ and consequently $\nu^{-1}(r) \in \Lambda(S)$, i.e., $r \in \nu[\Lambda(S)]$. Since $\nu[\Lambda(S)]$ is closed under superpositions, we have $\Lambda(\nu[S]) \subset \nu[\Lambda(S)]$. We can repeat the same reasoning for the symmetry (α^{-1}, ν^{-1}) and the set $\nu[S] \subset P$ instead of S. Thus we get $\Lambda(\nu^{-1}[\nu[S]]) \subset \nu^{-1}[\Lambda(\nu[S])]$, consequently $\nu[\Lambda(S)] \subset \Lambda(\nu[S])$.

Proposition 8. Let (α, ν) arise from a symmetry. Then if $S \subset P$ is a sector, $\nu[S]$ is a sector as well.

Proof. We have to show the properties (i)–(iii) from the definition of a sector. Property (i) follows from Proposition 7. To show (ii), let $p, q \in \nu[S]$, then $\nu^{-1}(p), \nu^{-1}(q) \in S$ and there is an $r \in S$ such that $r \in \Lambda(\{\nu^{-1}(p), \nu^{-1}(q)\})$, $r \neq \nu^{-1}(p), r \neq \nu^{-1}(q)$. Then as in the proof of Proposition 7, $\nu(r) \in \Lambda(\{p, q\})$, $\nu(r) \neq p, \nu(r) \neq q$. For (iii), let $q \in P, q \notin \nu[S]$. Let $p \in \nu[S]$, that is, $\nu^{-1}(p) \in S$. Then $\Lambda(\{\nu^{-1}(q), \nu^{-1}(p)\}) = \{\nu^{-1}(q), \nu^{-1}(p)\}$, and, again by Proposition 7, $\Lambda(\{p, q\}) = \{p, q\}$. □

Thus we have shown that symmetries permute the sectors. In the following proposition we shall suppose that P is the union of its sectors. We shall first prove a lemma.

Lemma 1. Let $C = \{a : a \leftrightarrow b$ for all $b \in L\}$ be the center of L. Let (α, ν) arise from a symmetry. Then $c \in C$ implies $\alpha(c) \in (C)$.

Proof. It follows from Corollary 2.

Proposition 9. Let $P = \cup S_t$, $t \in T$, where S_t are sectors and T is any set. Let (α, ν) arise from a symmetry. Then $\nu[S_t] = S_t$ for any t implies that $\alpha(c) = c$ for any element c in the center C of the logic L.

Proof. Let $p, q \in S_t$ for some t. From the proof of Theorem 3 in (Pulmannová, 1976) it follows that $p(c) = q(c)$ (and this equals 0 or 1) for all $c \in C$. Now let $p \in S_t$ imply $\nu(p) \in S_t$ for all $t \in T$. Then for any $c \in C$, $\nu(p)(c) = p(c)$ for all $c \in C$. As $\alpha(c) \in C$ provided $c \in C$, we get also $p(\alpha(c)) = \nu(p)(\alpha(c)) = p(c)$ for all $p \in P$. From this it follows that $P_{\alpha(c)} = P_c$, i.e., $\alpha(c) = c$.

The center C of a logic L is discrete if there exists an at most countable set $\{c_n\}_{n\in D}$ of mutually disjoint elements of C such that (i) $\vee_n c_n = 1$, (ii) C consists precisely of all the lattice sums $\vee_{n\in Z} c_n$, where Z is an arbitrary subset of D. The c_n's are atoms of C. If L is a logic with a discrete center, then it can be thought of as a direct sum of the irreducible logics $L_j = L_{[0,c_j]} = \{a \in L : a \leq c_j\}$ and $P = \cup P_j$, where P_j are disjoint subsets of P generated by pure states on L_j (Varadarajan, 1968).

Lemma 2. Let (L, M) be a quantum logic and L have a discrete center C. Let α arise from a symmetry; then c is an atom of C if and only if $\alpha(c)$ is an atom of C.

Proof. c is an atom of C if $d \leq c$, $d \in C$ implies $d = 0$ or $d = c$. As α, α^{-1} preserve order, from $d \leq \alpha(c)$ it follows that $\alpha^{-1}(d) \leq c$, that is, $\alpha^{-1}(d) = 0$ or $\alpha^{-1}(d) = c$, from which we get that $d = 0$ or $d = \alpha(c)$. The converse part can be proved analogously.

Proposition 10. Let (L, M) be a quantum logic and L have a discrete center C. Let (α, ν) be a symmetry of (L, M). Then $\alpha[L_{[0,c_j]}] = L_{[0,\alpha(c_j)]}$ and $\nu[P_j] = P_k$, where $c_k = \alpha(c_j)$, for all atoms $c_j \in C$.

Proof. Let $a \in L_{[0,c_j]}$, i.e., $a \leq c_j$. As α preserves the order, $\alpha(a) \leq \alpha(c_j)$, i.e., $\alpha(a) \in L_{[0,\alpha(c_j)]}$. Now let $\tilde{p} \in P_n$, then $\tilde{p}(a) = p(a \wedge c_n)$ for $a \in L$, where p is a pure state on $L_{[0,c_n]}$. Then $\tilde{p}(c_n) = p(c_n) = 1$ and for $m \neq n, \tilde{p}(c_m) = p(c_m \wedge c_n) = 0$. From this it follows that $\nu(\tilde{p})(\alpha(c_n)) = \tilde{p}(c_n) = 1$ and $\nu(\tilde{p})(\alpha(c_m)) = \tilde{p}(c_m) = 0$, so that $\nu(\tilde{p}) \in P_k$, where $c_k = \alpha(c_n)$.

Corrollary 3. If $\alpha(c) = c$ for all $c \in C$, then $\alpha[L_{[0,c_n]}] = L_{[0,c_n]}$ and $\nu[P_n] = P_n$.

If L has a discrete center and $P = \cup P_i$, where P_i are sectors, then the converse of Proposition 9 is also true.

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